

Improved Bounds on the Restricted Isometry Constant for Orthogonal Matching Pursuit

Jinming Wen, Xiaomei Zhu and Dongfang Li

In this letter, we first construct a counter example to show that for any given positive integer $K \geq 2$ and for any $\frac{1}{\sqrt{K+1}} \leq t < 1$, there always exist a K -sparse \mathbf{x} and a matrix \mathbf{A} with the restricted isometry constant $\delta_{K+1} = t$ such that the OMP algorithm fails in K iterations. Secondly, we show that even when $\delta_{K+1} = \frac{1}{\sqrt{K+1}}$, the OMP algorithm can also perfectly recover every K -sparse vector \mathbf{x} from $\mathbf{y} = \mathbf{A}\mathbf{x}$ in K iteration. This improves the best existing results which were independently given by Mo et al. and Wang et al.

Introduction: Consider the following linear model:

$$\mathbf{y} = \mathbf{A}\mathbf{x} \quad (1)$$

where $\mathbf{x} \in \mathbb{R}^n$ is an unknown signal, $\mathbf{y} \in \mathbb{R}^m$ is an observation vector and $\mathbf{A} \in \mathbb{R}^{m \times n}$ (with $m < n$) is a known sensing matrix. This model arises from compressed sensing, see, e.g., [5] and one of the central goals is to recover \mathbf{x} based on \mathbf{A} and \mathbf{y} . It has been shown that under some suitable conditions, \mathbf{x} can be recovered exactly, see, e.g., [2].

The orthogonal matching pursuit (OMP) [6] is one of the commonly used algorithms to recover \mathbf{x} from (1). A vector $\mathbf{x} \in \mathbb{R}^n$ is k -sparse if $|\text{supp}(\mathbf{x})| \leq k$, where $\text{supp}(\mathbf{x}) = \{i : x_i \neq 0\}$ is the support of \mathbf{x} . For any set $T \subset \{1, 2, \dots, n\}$, let \mathbf{A}_T be the submatrix of \mathbf{A} that only contains columns indexed by T and \mathbf{x}_T be the restriction of the vector \mathbf{x} to the elements indexed by T . Then the OMP can be described by Algorithm 1.

One of the commonly used frameworks for sparse recovery is the restricted isometry property, which was introduced in [2]. For any $m \times n$ matrix \mathbf{A} and any integer $k, 1 \leq k \leq n$, the k -restricted isometry constant δ_k is defined as the smallest constant such that

$$(1 - \delta_k) \|\mathbf{x}\|_2^2 \leq \|\mathbf{A}\mathbf{x}\|_2^2 \leq (1 + \delta_k) \|\mathbf{x}\|_2^2 \quad (2)$$

for all k -sparse vector \mathbf{x} .

It has conjectured in [3] that there exist a matrix with $\delta_{K+1} \leq \frac{1}{\sqrt{K}}$ and a K -sparse \mathbf{x} such that the OMP fails in K iterations [8]. Counter examples have independently given in [8] and [9] that there exist a matrix with $\delta_{K+1} = \frac{1}{\sqrt{K}}$ and a K -sparse \mathbf{x} such that the OMP fails in K iterations. In this letter, we will give a counter example to show that for any given positive integer $K \geq 2$ and for any $\frac{1}{\sqrt{K+1}} \leq t < 1$, there always exist a K -sparse \mathbf{x} and a matrix \mathbf{A} with $\delta_{K+1} = t$ such that the OMP algorithm fails in K iterations. This result not only greatly improves the existing results, but also gives a counter example with $\delta_{K+1} < \frac{1}{\sqrt{K}}$ such that the OMP fails in K iterations.

It has respectively shown in [4] and [7] that $\delta_{K+1} < \frac{1}{3\sqrt{K}}$ and $\delta_{K+1} < \frac{1}{(1+\sqrt{2})\sqrt{K}}$ are sufficient for OMP to recover every K -sparse \mathbf{x} in K iteration. The sufficient condition has independently improved to $\delta_{K+1} < \frac{1}{1+\sqrt{K}}$ in [8] and [9]. In this letter, we will improve it to $\delta_{K+1} \leq \frac{1}{1+\sqrt{K}}$.

Algorithm 1 OMP [6], [9]

Input: measurements \mathbf{y} , sensing matrix \mathbf{A} and sparsity K .

Initialize: $k = 0, \mathbf{r}^0 = \mathbf{y}, T^0 = \emptyset$.

While $k < K$

- 1: $k = k + 1$,
- 2: $t^k = \arg \max_j |\langle \mathbf{r}^{k-1}, \mathbf{A}_j \rangle|$,
- 3: $T^k = T^{k-1} \cup \{t^k\}$,
- 4: $\hat{\mathbf{x}}_{T^k} = \arg \min_{\mathbf{x}} \|\mathbf{y} - \mathbf{A}_{T^k} \mathbf{x}\|_2$,
- 5: $\mathbf{r}^k = \mathbf{y} - \mathbf{A}_{T^k} \hat{\mathbf{x}}_{T^k}$.

Output: $\hat{\mathbf{x}} = \arg \min_{\mathbf{x}: \text{supp}(\mathbf{x})=T^K} \|\mathbf{y} - \mathbf{A}\mathbf{x}\|_2$.

Main Results: In this section, we will give our main results. We will first construct a counter example to show the OMP algorithm may fail in K iterations if $\frac{1}{\sqrt{K+1}} \leq \delta_{K+1} < 1$.

Theorem 1: For any given positive integer $K \geq 2$ and for any

$$\frac{1}{\sqrt{K+1}} \leq t < 1$$

there always exist a K -sparse \mathbf{x} and a matrix \mathbf{A} with the restricted isometry constant $\delta_{K+1} = t$ such that the OMP fails in K iterations.

Our proof is similar to the method used in [8], but the critical idea is different.

Proof. For any given positive integer $K \geq 2$, let

$$\mathbf{B} = \begin{bmatrix} \frac{K}{K+1} \mathbf{I}_K & \frac{1}{K+1} \\ \frac{1}{K+1}^T & \frac{K+2}{K+1} \end{bmatrix}$$

where $\mathbf{1}$ is a K -dimensional column vector with all of its entries being 1 and \mathbf{I}_K is the K -dimensional identity matrix.

By some simple calculations, we can show that the eigenvalues $\{\lambda_i\}_{i=1}^{K+1}$ of \mathbf{B} are

$$\lambda_1 = \dots = \lambda_{K-1} = \frac{K}{K+1}, \lambda_K = 1 - \frac{1}{\sqrt{K+1}}, \lambda_{K+1} = 1 + \frac{1}{\sqrt{K+1}}.$$

let $s = t - \frac{1}{\sqrt{K+1}}$ and $\mathbf{C} = \mathbf{B} - s\mathbf{I}_{K+1}$. Then by the aforementioned two equations, the eigenvalues $\{\lambda_i\}_{i=1}^{K+1}$ of \mathbf{C} are

$$\lambda_1 = \dots = \lambda_{K-1} = \frac{K}{K+1} - s$$

$$\lambda_K = 1 - \frac{1}{\sqrt{K+1}} - s = 1 - t, \lambda_{K+1} = 1 + \frac{1}{\sqrt{K+1}} - s.$$

Since $\frac{1}{\sqrt{K+1}} \leq t < 1$, \mathbf{C} is a symmetric positive definite matrix. Therefore, there exists an upper triangular matrix \mathbf{A} such that $\mathbf{A}^T \mathbf{A} = \mathbf{C}$. By the aforementioned inequations and (2), $\delta_{K+1}(\mathbf{A}) = t$.

Let $\mathbf{x} = (1, 1, \dots, 1, 0) \in \mathbb{R}^{K+1}$, then \mathbf{x} is K -sparse. Let $\mathbf{e}_i, 1 \leq i \leq K+1$, denote the i -th column of \mathbf{I}_{K+1} , then one can easily show that,

$$\frac{K}{K+1} - s = \max_{1 \leq i \leq K} |\langle \mathbf{A}\mathbf{e}_i, \mathbf{A}\mathbf{x} \rangle| \leq |\langle \mathbf{A}\mathbf{e}_{K+1}, \mathbf{A}\mathbf{x} \rangle| = \frac{K}{K+1}$$

so the OMP fails in the first iteration. Therefore, the OMP algorithm fails in K iterations for the given vector \mathbf{x} and the given matrix \mathbf{A} .

In the following, we will improve the sufficient condition $\delta_{K+1} < \frac{1}{\sqrt{K+1}}$ [8], [9] of the perfect recovery to $\delta_{K+1} \leq \frac{1}{\sqrt{K+1}}$.

Theorem 2: Suppose that \mathbf{A} satisfies the restricted isometry property of order $K+1$ with the restricted isometry constant

$$\delta_{K+1} = \frac{1}{\sqrt{K+1}} \quad (3)$$

then the OMP algorithm can perfectly recover any K -sparse signal \mathbf{x} from $\mathbf{y} = \mathbf{A}\mathbf{x}$ in K iteration.

Before proving this theorem, we need to introduce the following two lemmas, where Lemma 2 was proposed in [9].

Lemma 1: For each \mathbf{x}, \mathbf{x}' supported on disjoint subsets $S, S' \subseteq \{1, \dots, n\}$ with $|S| \leq s, |S'| \leq s'$, we have

$$|\langle \mathbf{A}\mathbf{x}, \mathbf{A}\mathbf{x}' \rangle| = \delta_{s+s'} \|\mathbf{x}\|_2 \|\mathbf{x}'\|_2 \quad (4)$$

if and only if:

$$\frac{\|\mathbf{A}\mathbf{x}\|_2^2}{\|\mathbf{x}\|_2^2} + \frac{\|\mathbf{A}\mathbf{x}'\|_2^2}{\|\mathbf{x}'\|_2^2} = 2. \quad (5)$$

Proof. Let

$$\bar{\mathbf{x}} = \mathbf{x} / \|\mathbf{x}\|_2, \bar{\mathbf{x}}' = \mathbf{x}' / \|\mathbf{x}'\|_2 \quad (6)$$

since $S \cap S' = \emptyset$, we have, $\|\bar{\mathbf{x}} + \bar{\mathbf{x}}'\|_2^2 = \|\bar{\mathbf{x}} - \bar{\mathbf{x}}'\|_2^2 = 2$. By (2), we have

$$2(1 - \delta_{s+s'}) \leq \|\mathbf{A}(\bar{\mathbf{x}} \pm \bar{\mathbf{x}}')\|_2^2 \leq 2(1 + \delta_{s+s'}) \quad (7)$$

By the parallelogram identity and (6), we have

$$|\langle \mathbf{A}\bar{\mathbf{x}}, \mathbf{A}\bar{\mathbf{x}}' \rangle| = \frac{1}{4} \|\mathbf{A}(\bar{\mathbf{x}} + \bar{\mathbf{x}}')\|_2^2 - \|\mathbf{A}(\bar{\mathbf{x}} - \bar{\mathbf{x}}')\|_2^2 \leq \delta_{s+s'}. \quad (8)$$

By (6), (4) holds if and only if the equality in (8) holds. By (7), the equality in (8) holds if and only if

$$\|\mathbf{A}(\bar{\mathbf{x}} + \bar{\mathbf{x}}')\|_2^2 = 2(1 - \delta_{s+s'}), \|\mathbf{A}(\bar{\mathbf{x}} - \bar{\mathbf{x}}')\|_2^2 = 2(1 + \delta_{s+s'})$$

or

$$\|\mathbf{A}(\bar{\mathbf{x}} - \bar{\mathbf{x}}')\|_2^2 = 2(1 - \delta_{s+s'}), \|\mathbf{A}(\bar{\mathbf{x}} + \bar{\mathbf{x}}')\|_2^2 = 2(1 + \delta_{s+s'}).$$

Therefore, (4) holds if and only if

$$\|\mathbf{A}(\bar{\mathbf{x}} - \bar{\mathbf{x}}')\|_2^2 + \|\mathbf{A}(\bar{\mathbf{x}} + \bar{\mathbf{x}}')\|_2^2 = 4.$$

Obviously, the aforementioned equation is equivalent to (5), so the lemma holds.

Lemma 2: For $S \subset \{1, 2, \dots, n\}$, if $\delta_{|S|} < 1$, then

$$(1 - \delta_{|S|}) \|\mathbf{x}\|_2 \leq \|\mathbf{A}_S^T \mathbf{A}_S \mathbf{x}\|_2 \leq (1 + \delta_{|S|}) \|\mathbf{x}\|_2$$

for any vector \mathbf{x} supported on S .

We will prove it by induction. Our proof is similar to the method used in [8], but the critical idea is different.

Proof of Theorem 2 Firstly, we prove that if (3) holds, then the OMP can choose a correct index in the first iteration.

Let S denote the support of the K -sparse signal \mathbf{x} and let $\alpha = \max_{i \in S} |\langle \mathbf{A} \mathbf{e}_i, \mathbf{A} \mathbf{x} \rangle|$. Then

$$|\langle \mathbf{A} \mathbf{x}, \mathbf{A} \mathbf{x} \rangle| = \left| \sum_{i \in S} x_i \langle \mathbf{A} \mathbf{e}_i, \mathbf{A} \mathbf{x} \rangle \right| \leq \alpha \|\mathbf{x}\|_1 \leq \alpha \sqrt{K} \|\mathbf{x}\|_2.$$

By (2), it holds that

$$|\langle \mathbf{A} \mathbf{x}, \mathbf{A} \mathbf{x} \rangle| \geq (1 - \delta_{K+1}) \|\mathbf{x}\|_2^2. \quad (9)$$

By the aforementioned two inequations, we have

$$\max_{i \in S} |\langle \mathbf{A} \mathbf{e}_i, \mathbf{A} \mathbf{x} \rangle| \geq \frac{(1 - \delta_{K+1}) \|\mathbf{x}\|_2}{\sqrt{K}} \quad (10)$$

and if the equality in (10) holds, then the equality in (9) must also hold.

By Lemma 2.1 in [1], for each $j \notin S$, it holds

$$|\langle \mathbf{A} \mathbf{e}_j, \mathbf{A} \mathbf{x} \rangle| \leq \delta_{K+1} \|\mathbf{x}\|_2. \quad (11)$$

So if (3) holds and at least there is one equality in (9) or (11) does not hold, then for each $j \notin S$, it holds

$$|\langle \mathbf{A} \mathbf{e}_j, \mathbf{A} \mathbf{x} \rangle| < \max_{i \in S} |\langle \mathbf{A} \mathbf{e}_i, \mathbf{A} \mathbf{x} \rangle|.$$

Therefore, it suffices to show that the equality in (9) and the equation in (11) can not hold simultaneously.

Suppose both the equality in (9) and the equation in (11) hold, then by Lemma 1, $\|\mathbf{A} \mathbf{e}_j\|_2^2 = 1 + \delta_{K+1}$. Let $\mathbf{C} = (\mathbf{A}_{S \cup j})^T \mathbf{A}_{S \cup j}$, then $C_{jj} = \|\mathbf{A} \mathbf{e}_j\|_2^2 = 1 + \delta_{K+1}$, thus for each $i \in S$, $C_{ij} = 0$. In fact, suppose there exists one $i \in S$ such that $C_{ij} \neq 0$, then

$$\|\mathbf{A}_{S \cup j}^T \mathbf{A}_{S \cup j} \mathbf{e}_j\|_2 \geq \sqrt{\sum_{i \in S} C_{ij}^2} > 1 + \delta_{K+1}$$

which contradicts Lemma 2. Therefore, for each $i \in S$, $C_{ij} = 0$. However, in this case, we have

$$|\langle \mathbf{A} \mathbf{e}_j, \mathbf{A} \mathbf{x} \rangle| = 0$$

which contradicts the equality in (11). Thus the equality in (9) and the equation in (11) can not hold simultaneously. Therefore, if (3) holds, then the OMP can choose a correct index in the first iteration.

By applying the method used in [8] or [9] and the aforementioned proof, one can similarly show that if (3) holds, then the OMP can choose a correct index in the latter iterations, so the theorem is proved.

Future Work: In the future, we will prove or disprove whether $\frac{1}{\sqrt{K+1}} < \delta_{K+1} < \frac{1}{\sqrt{K+1}}$ is a sufficient condition for the OMP to recover every K -spares signal \mathbf{x} from $\mathbf{y} = \mathbf{A} \mathbf{x}$ in K iterations.

Acknowledgment: This work has been supported by NSFC (Grant No. 11201161, 11171125, 91130003) and FRQNT.

Jinming Wen (Dept. of Mathematics and Statistics, McGill University, Canada, H3A 2K6)

Xiaomei Zhu (College of Electronics and Information Engineering, Nanjing University of Technology, China, 211816; Dept. of Electrical and Computer Engineering, McGill University, Canada, H3A 2A7)

E-mail: njiczm@njut.edu.cn

Dongfang Li (School of Mathematics and Statistics, Huazhong University of Science and Technology, China, 430074; Dept. of Mathematics and Statistics, McGill University, Canada, H3A 2K6)

References

- 1 Candés, E. J.: 'The restricted isometry property and its implications for compressed sensing', *Compte Rendus de l'Academie des Sciences, Paris, Serie I*, 2008, **346**, pp. 589–592
- 2 Candés, E. J. and Tao, T.: 'Decoding by linear programming', *IEEE Trans. Inf. Theory*, 2005, **51**, pp. 4203–4215
- 3 Dai, W. and Milenkovic, O.: 'Analysis of Orthogonal Matching Pursuit Using the Restricted Isometry Property', *IEEE Trans. Inf. Theory*, 2009, **55**, pp. 2230–2249
- 4 Davenport, M. A. and Wakin, M. B.: 'Analysis of Orthogonal Matching Pursuit Using the Restricted Isometry Property', *IEEE Trans. Inf. Theory*, 2010, **56**, pp. 4395–4401
- 5 Donoho, D. L.: 'Compressed sensing', *IEEE Trans. Inf. Theory*, 2006, **52**, pp. 1289–1306
- 6 Tropp, J. A. and Gilbert, A. C.: 'Signal Recovery From Random Measurements Via Orthogonal Matching Pursuit', *IEEE Trans. Inf. Theory*, 2007, **53**, pp. 4655–4666
- 7 Liu, E. and Temlyakov, V. N.: 'The Orthogonal Super Greedy Algorithm and Applications in Compressed Sensing', *IEEE Trans. Inf. Theory*, 2012, **58**, pp. 2040–2047
- 8 Mo, Q. and Shen, Y.: 'A Remark on the Restricted Isometry Property in Orthogonal Matching Pursuit', *IEEE Trans. Inf. Theory*, 2012, **58**, pp. 3654–3656
- 9 Wang, J. and Shim, B.: 'On the Recovery Limit of Sparse Signals Using Orthogonal Matching Pursuit', *IEEE Trans. Signal Process.*, 2012, **60**, pp. 4973–4976